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Group Choice with Interdependent Sublinear Voting

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Summary

Many important decisions of policy are made in a collective manner, so a great deal of formal and informal analysis has been devoted to collective decision-making processes (CDMPs). One common approach to analyzing CDMPs is first to model the participants as exercising their ‘votes’ independently, and secondly to solve the model and identify a range of feasible outcomes. A potential criticism of this approach is that the actors may strategically modify their behaviors based on the behaviors of others. If such interdependent voting behaviors led to different outcomes, it would undermine the validity of analysis based on independent voting.

In this paper, we compare and contrast two models of collective choice. The first assumes that each participant decides how much effort to exert as if they were acting alone; the second assumes that each participant strategically takes into account the effort exerted by other participants. While the first model is simpler, the second is more realistic.

This paper demonstrates that they produce the same collective outcome: the simple model can be used without sacrificing realism.

More formally, we demonstrate the mathematical equivalence of these two different CDMPs:

> The simple strategy of independent proportional voting, and some simple results of applying it.

> The consequences when actors strategically modify their behavior to take into account each other’s actions.

There are some subtle differences in how the ‘negotiations’ play out — the actors do tend to exert less effort because of free rider effects — but overall the modified CDMP gives the same result.

Thus, we can analyze CDMPs as if the actors used independent proportional voting and remain confident that the final result is also correct should actors be using more strategically sophisticated behavior.

Introduction

KAPSARC has outlined an analytical approach to assist in understanding the likely range of potential outcomes of collective decision making processes. This is described in *An Introduction to the KAPSARC Toolkit for Behavioral Analysis (KTAB)* and *Multidimensional Bargaining Using KTAB* (Wise, Lester and Efird 2015).

The analytical approach simply formalizes common sense notions. In almost every collective decision-making process, people cluster into interest groups which we term 'actors'. These might be formal or informal.

> **Formal**: political parties, industry associations, etc.

> **Informal**: subsistence farmers, patronage networks, linguistic groups, etc.

These groups have common interests; this can be taken as the operational definition of an actor in the sense that subgroups with strongly opposing interests should be considered as separate, opposing actors. For example, subsistence farmers might have a strong interest in subsidized water, electricity, or staple foods. A patronage network might have an interest in increased opportunities for the leading members to acquire wealth so that it can be redistributed downward.

These actors exert influence. The influence may be exerted directly, such as bloc voting by legislators, or quite indirectly. Formal voting in committees, either 'one person, one vote' or weighted, is the most familiar kind of voting. But there are many different ways and means to 'weigh in' and exert influence. Public interest groups can allocate a large quantity of money to public information campaigns on issues they consider important, but very little to the unimportant ones. In formal committees, interest groups exert informal influence by letting politicians know which policies they support or oppose – and how the politician's behavior will affect the campaign contributions they are likely to receive in the future. The size and direction of those contributions constitute a kind of 'generalized vote' for or against various policies. Even in a military dictatorship, the dictator must consider how policy choices will affect future support by the military, so the military factions are again exerting influence, which we consider a kind of generalized vote.

Actors exert little effort when they perceive that there is little at stake. For example, labor unions do not expend their entire strike fund whenever there is a small disagreement – unless that is an excuse to act on an important issue. However, when the stakes are 'life or death', then it can be expected that the groups will exert great effort.

The actors exert influence to promote one outcome over another. For example, different patronage groups may promote different tax and trade policies. Strategically, policies will be crafted to balance two considerations:

> **Self-Interest**: policies might be designed to bring revenue into the industries to which a patronage group is connected.

> **Plausible**: policies might be designed to bring benefits to other actors, winning enough support from other actors to plausibly prevail.

Finally, actors that exert more influence tend to prevail over actors that exert influence; this could be taken as an operational definition of 'more influence'. Even in consensus-based systems, some actors achieve more of what they want and others achieve less. Note that the decisive factor is the influence actually exerted in a particular situation, as opposed to the maximum which might theoretically be exerted.

The linked ideas of actors, choices and influence form a high-level conceptual framework similar to the supply, demand, and price framework of economics. It is very important to understand at what level of analysis this framework is aimed. In economics, the concepts of supply/demand curves and of production functions are ubiquitous. However, these concepts are useful for systematic analysis from ‘outside’ the system. For example, designing the layout of an assembly line for mobile phones is a problem for engineering, not economics. In a sense, economic theory is designed to characterize the behavior of the system ‘from the outside’, not to analyze, design or control its components ‘from the inside’. Similarly, KTAB is intended to characterize the interactions of actors, choices, and influence ‘from the outside’.

Within the actor, choice and influence framework, the initial papers explored simple models of behavior that are found in the literature and explained the algorithms behind the tools that have been released at the KTAB webpage. This paper takes the analysis further and demonstrates that a more sophisticated class of CDMPs can also be analyzed using the tools that we have developed.

As we have already explained, the building blocks of a CDMP are:

- The set of actors involved.
- The set of options among which they are deciding.
- How the actors value the options and the extent to which they wish/are able to influence the outcome to secure the perceived value.

The differing approaches to analysis are based on the way in which these building blocks are assembled.

Collective Decision-Making With Generalized Voting

For this class of CDMP, each actor represents not a single person but a coherent interest group, such as 'subsistence farmers' or 'aerospace manufacturers'; individual leaders are taken as representatives of the groups they lead. Actors exert influence to promote their preferred options, so an essential feature of each actor is the maximum influence it could bring to bear to sway the debate, if it were motivated to do so. The influence may be in terms of formal ballots, informal persuasion before a formal vote, military intimidation, economic inducements, or whatever form of influence is appropriate to the situation being analyzed.

The set of possible options is designated Ω. An option θ might be a particular policy, a particular bargain between two or three other actors over their respective positions, allocation of disputed territories to nations, the choice to back one or the other party in a conflict, and so on. Every option must fall in the set of possible options: $\theta \in \Omega$.

For any particular option, each of the N actors has a numeric utility function that describes the attractiveness, to them, of the consequences they expect from adopting that policy. We generally use the von Neumann normalization (0,1), and rule out interpersonal comparison of utility.

In noncooperative game theory (NCGT), each actor chooses its actions independently, where the payoff to each depends on not only its circumstances and choice but also on the actions of all actors. A strategy is a complete specification of what an actor will do in every circumstance. The act of exerting influence to promote one option (a) over another (b) is called generalized voting. Hence, the strategy of actor i is its rule for determining how much influence to exert when faced with a choice of two alternatives. It is summarized in its generalized voting function, $v_i\,(a{:}b)$. Positive values favor a and negative values favor b.

When the means of exerting influence is casting a single yes/no ballot, aka 'one person, one vote' (1P1V), the vote between a and b is +1 if the actor voted for a, -1 if it voted for b, and 0 otherwise. As discussed in *Multidimensional Bargaining Using KTAB* (Wise, Lester and Efird 2015), this generalized voting notation can be used to represent many voting systems, such as approval voting, range voting, Borda voting, super majority rule, best two win, etc.

In this notation, the coalition supporting a over b is simply the set of actors which exert influence supporting a over b:

$$c(a{:}b) = \{i \mid v_i\,(a{:}b) > 0\}$$

Equation 1: Members of a coalition

The strength of the coalition supporting a over b is the sum of their votes:

$$s(a{:}b) = \sum_{i \in c(a:b)} v_i(a{:}b)$$

Equation 2: Strength of a coalition

In the 1P1V case, $s(a,b)$ is just the number who voted for a.

The net group influence is just the sum of individual votes:

$$V(a{:}b) = \sum_{i=1}^{N} v_i(a{:}b)$$

Equation 3: Net vote is sum of individual votes

In the 1P1V case, V is simply the difference in total votes for a and for b. Because $s(a{:}b)$ collects all the terms favoring a and $s(b{:}a)$ collects all the terms favoring b, it follows that the net group influence is just the difference between the strength of coalitions supporting each option:

$$V(a{:}b) = s(a{:}b) - s(b{:}a)$$

Equation 4: Net vote is difference between coalition strengths

This defines a relationship between options based on group dominance. Option b is displaced by a, or a is preferred to b if, and only if, the net group influence favors a over b:

$$b \rightarrow a \;\; \mathit{iff} \;\; V(a{:}b) > 0$$

Equation 5: Preference ordering determined by net vote

An expression like $w \rightarrow x \rightarrow y \rightarrow z$ is called a sequence, where each option is dominated by the following one. Note that this dominance relationship is not necessarily transitive, so $w \rightarrow x$ and $x \rightarrow y$ do not always imply $w \rightarrow y$. It is perfectly possible to have nontransitive, cyclic preferences, e.g. $w \rightarrow x \rightarrow y \rightarrow w$ or even $x \rightarrow y \rightarrow x$.

If one option dominates all others in a set, Ω, then it is called the Condorcet winner (CW) of that set, for those actors. Because $\theta \rightarrow \theta_{cw}$ for all θ, the existence of a CW requires that every sequence which includes θ_{cw} must terminate at θ_{cw}. But the behavior of sequences which do not include θ_{cw}, and the behavior before reaching θ_{cw} for those that do include it, are completely unconstrained.

The case of majority rule voting where each actor has acyclic preferences has been extensively analyzed since at least Condorcet, (1785). As is well known, the sequences typically have cycles unless special conditions are imposed, as the table below shows. Actors X, Y, and Z rank options A, B, and C from best at the top to worst at the bottom.

The group chooses C over B by a 2:1 margin, chooses B over A, and chooses A over C. Thus, the sequence $A \rightarrow B \rightarrow C \rightarrow A$ is cyclic, and 1P1V yields no clear choice.

The larger the set Ω, the more likely that a cyclic sequence will reach every option. As stated in Coughlin (1992):

"One final (very significant) property of majority rule that arises when cycles exist is that for any pair of alternatives, it is (usually) possible to find a sequence … that starts at the first alternative in the pair and ends at the second. This tells us that when cycles exist, it is (usually) possible for majority-rule sequences to wander all over the set of alternatives."

Table 1. Cyclic preferences under majority voting

X	Y	Z
C	A	B
B	C	A
A	B	C

Source: KAPSARC

These results are not changed when actors have different voting weights, w_i, as is common in share-weighted voting in corporate meetings, or committees where members vote as blocs. This has led many to conclude that, in theory, organizations under majority rule will transition between policies with no particular direction and certainly without stability. Though individual sequences are not totally unstructured, the theory seems to suggest that for any two arbitrary options, X and Y, it is usually the case that majority rule will eventually reach X, then eventually reach Y, then continue wandering.

Coughlin (1992) provides an excellent overview of the literature, including this summary of the situation:

“The results described so far have led some public-choice scholars to conclude that, except in (empirically) rare cases, no matter what policies are initially selected, they are (almost always) replaced by alternative policies in the next election or the next time the deciding legislature or committee considers them. This broad inference has been best summarized by Riker (1982a,19) in his assertion that ‘what we have learned is simply this: Disequilibrium, or the potential that the status quo be upset, is the characteristic feature of politics’”.

Yet, in practice, organizations which employ voting often exhibit remarkable consistency and stability. As stated in Coughlin (1992, 9):

“Rather, as Tullock (1981a) has argued, in contrast to what has been shown to be true about the majority-rule relation, government policies do not tend to change quickly or to wander all over the set of possible policies. As a consequence, it is hard to resist the alternative inference that the primary contribution of recent work on the majority-rule relation is as a reductio ad absurdum that tells us to reconsider the basic model to see how it should be modified so that theory and empirical observations match up. As Romer and Rosenthal (1984) put it: ‘The public policies of most mature democracies, however, do not reflect the kind of indeterminate cycling that this instability suggests. Instead, we see that public policies and the composition of governments are fairly stable. … This … suggests that research on voting models should focus on the aspects of the political process … that somehow “solve” the instability problem.’ “

Therefore, analysis of voting has typically focused on one of two approaches:

> Institutional arrangements. Attention initially focused on institutions that prevented cycling by constraining which options are considered and by shaping preferences; it has since expanded to analyze institutions more generally. Public choice theory and mechanism design both use this approach extensively.

> Special conditions where the dominance sequences are provably acyclic.

Discrete Voting

The above approaches assume discrete yes/no votes. In the case of formal committee votes, each actor has one indivisible ballot, so the only possible behavior is discrete yes/no voting. Formally, each exerts the maximum influence of which it is capable for whichever option it prefers, however slight is that preference. This binary, all-or-nothing behavior can be represented in a formula as follows:

$$v_i(a\colon b) = \begin{cases} +w_i & \text{if } u_i(a) > u_i(b) \\ -w_i & \text{if } u_i(a) < u_i(b) \end{cases}$$

Equation 6: Binary voting rule

Notice that, for any individual with acyclic binary voting, we can construct an infinite number of real-valued utility functions which are consistent with that voting pattern, as follows. If the set Ω of alternatives is finite, then we can simply sort the options by that actor's voting:

$$\theta_1 \rightarrow \theta_2 \rightarrow \theta_3 \rightarrow \cdots \rightarrow \theta_M$$

Equation 7: Acyclic preference order

Then we can choose any arbitrary ascending sequence of numbers, a_i, and define the corresponding utility function.

$$0 \leq a_1 < a_2 < a_3 < \cdots < a_M \leq 1$$

$$u(\theta_i) = a_i$$

Equation 8: Utilities consistent with acyclic preference

Under certain technical assumptions regarding limits and closed sets, these results generalize to countably infinite and even uncountably infinite, or continuous, sets of options, Ω.

A common requirement for voting models is that the utility functions of every actor must be single-peaked. This can only be meaningfully defined where there is also a distance measure, d, so that we can say that the utility function not only has a maximum at a given point θ^* but declines with increasing distance from θ^*:

$$d(\theta^*, \theta_2) > d(\theta^*, \theta_1) \Rightarrow u(\theta_2) < u(\theta_1)$$

Equation 9: Single-peakedness

When the set of options can be described as a continuous N-dimensional set of real numbers, $\Omega \subseteq R^N$, then some version of weighted Euclidean distance is commonly used for a distance measure. When the set of options is inherently discrete, then the concept of 'single-peaked' must be addressed on a case-by-case basis, depending on whether or not there is a useful definition of distance for the problem at hand which could be applied in Equation 9.

Black (1948) showed that under certain conditions, the binary voting rule led to a CW. This result is the famous Median Voter Theorem (MVT). The essential conditions are that Ω=[0,1], every actor has a bounded and single-peaked utility function, and the strategy of every actor is the binary voting rule, Equation 6. The MVT's assumptions meet the requirements for limits and closed sets over an uncountable set. As is well known, the MVT does not generalize to multiple dimensions, multi-peaked utilities, or discrete options in any simple way.

Continuous Voting

As described in Coughlin (1992), many authors have concluded that the basic model of binary voting leads to absurdly incorrect results and must be fundamentally revised to solve the instability problem. One such revision is to replace binary votes with continuous votes.

In committees or legislatures where members can vote as blocs, the only rational strategy for them is to use binary voting. Again, no matter how slight their preference between options, it is only rational for the voting blocs to exert their maximum influence, because it maximizes their probability of success, and there are no costs to the act of exerting influence – voting – per se. The latter point becomes crucial in the case of informal influence, where there are often significant costs to exerting influence.

When the mere act of exerting influence has costs, we often observe much more nuanced application of influence. In the case of exerting military power, the costs of exerting influence include the loss of deterrence in other theaters, high financial cost of deployments, deaths of uninvolved civilians, proportional retaliation and the ever-present risk of escalation. Obviously, the great military powers of the world do not unleash their full arsenal at the slightest provocation, and interest groups do not expend all their lobbying funds on the first issue that arises no matter how trivial. The influence exerted tends to be small for low stakes, and large only for higher stakes.

Independent Proportional Voting

A heuristic approximation to this nuanced behavior is that each actor exerts influence proportional to its stakes in the choice:

$$v_i\,(a{:}b) = w_i\,(u_i\,(a) - u_i\,(b))$$

Equation 10: Proportional voting

Each actor is myopic in that it chooses how much influence to exert without reference to the behavior of other actors: its response is linear in the stakes for that particular actor. Essentially, it exerts influence as if it were 'going it alone'. We term this independent, proportional voting (IPV). As discussed earlier, this equation does not apply to discrete voting in committees, because it assumes the possibility of gradually adjusting the degree of influence, as opposed to a binary yes/no response.

Examples of Proportional Voting

Much of mathematics can be mapped onto the real world in several ways. We will discuss two possible applications of Equation 10.

One special case is the situation where actors exert some kind of informal influence, such as backroom persuasion that shapes the formal committee vote, deploying military forces to intimidate one nation or support another, expenditure of funds to shift public awareness and similar. In such informal cases, there is by definition no formal rule requiring a discrete vote, and actors have many ways to exert influence, each with their own costs and benefits. In this case, Equation 10 describes a deterministic situation where generalized vote is determined by the difference in utilities (though the outcome could still be stochastic, given those votes).

Another special case described by Equation 10 is a particular probabilistic model of 1P1V. In this case, Equation 10 gives only the expected value of stochastic voting. As will be discussed below, this is closely related to the case analyzed in Coughlin (1992). The group sequencing relationship → is deterministic, so we introduce the notation $P_i\,(a{:}b)$ for the probability that individual actor i will vote for a, given the choice between a and b. If all $u_i\,(\theta)\in[0,1]$ and all w_i=1, then the generalized vote is always between +1 and –1. Because probabilities are always between 0 and 1, their difference is always between +1 and -1. Thus, the proportional voting model can be seen as a particular model of the probability that actor i will vote for option a minus its probability of voting for b:

$$v_i\,(a{:}b) = P_i\,[a{:}b] - P_i\,[b{:}a]$$

Equation 11: Expected vote is difference in probabilities

In this interpretation, $V(a{:}b)$ is just the expected number of votes for a minus the expected number of votes for b. We add a superscript 'u' to the probabilities, to indicate that they are derived from a utility function. While we have great flexibility in choosing the u_i function for each actor, the pairwise probabilities P_i^u thus determined for each actor are not completely arbitrary because they are linked to the utility functions:

$$P_i^u[a{:}b] = \frac{1 + u_i(a) - u_i(b)}{2}$$

Equation 12: Probability determined by utilities

Binary Luce Voting Model

First, the set of options, Ω, is required to be a convex, compact subset of R^N.

Second, when the i-th actor is confronted with a choice between two options, a and b, the probability of choosing either is determined by the ratio of a function $f_i > 0$. We add a superscript 'L' to the probabilities to indicate that they are derived from the Luce model, not a utility function. Larger f-values are favored:

$$P_i^L(a:b) = \frac{f_i(a)}{f_i(a) + f_i(b)}$$

Equation 13: Probability from f-ratio

Similarly,

$$P_i^L(b:a) = \frac{f_i(b)}{f_i(b) + f_i(a)}$$

Equation 14: Converse probability from f-ratio

While we have great flexibility in choosing the f_i function for each actor, the pairwise probabilities P^L for each actor are not completely arbitrary because they are linked to the f_i functions.

Notice that the f_i functions are completely unrelated to the u_i functions. One symptom of this is that they obey different constraints: $0 < f_i$ versus $0 \leq u_i \leq 1$.

Third, every actor's f_i function is concave, is continuous and has continuous first derivative over all of Ω.

Fourth, each actor has a positive weight, w_i, which specifies the size of its vote. It casts its vote with probabilities as above, so the expected vote for a over b is the following:

$$\begin{aligned} v_i(a:b) &= w_i\left(P_i^L(a:b) - P_i^L(b:a)\right) \\ &= w_i\frac{f_i(a) - f_i(b)}{f_i(a) + f_i(b)} \end{aligned}$$

Equation 15: Expected vote is weighted difference in f-ratios

Thus, the expected margin of victory for option a over option b is the following:

$$V(a:b) = \sum_i v_i(a:b)$$

Equation 16: Expected net vote is sum of expected individual vote

Expanding terms,

$$V(a:b) = \sum_i w_i\frac{f_i(a) - f_i(b)}{f_i(a) + f_i(b)}$$

Equation 17: Expected vote is sum of weighted difference in f-ratios

This is clearly a different equation from IPV; we will shortly demonstrate that it corresponds to a fundamentally different CDMP.

As described in Coughlin (1992), we can define the utility function $U_i(a) = \ln f_i(a)$ which inherits the requirements of being concave, continuous and with continuous first derivative over all of Ω. It is used to define the following function:

$$W_B(x) = \sum_i w_i U_i(x)$$

Equation 18: Weighted sum of utilities

This functional form has been labeled a "Benthamite Social Utility Function" in Arrow (1963), though the relationship to Bentham's philosophy of utilitarianism is rather tenuous. The maximum of this function was also identified as the political equilibrium, under a different set of restrictive assumptions, in Becker (1983).

The BLVM covers the probabilistic choice model considered in Coughlin (1992), leading to his corollary 4.4, which states that the electoral equilibrium, i.e. the CW, is that position which maximizes the implicit Benthamite social utility function; for brevity, we call that position simply the Central Position (CP).

Note that Coughlin's corollary 4.4 initially uses a probabilistic voting model, but then assumes the actual votes are exactly the mean expected vote in order to pick out a deterministic CW. It would be more precise to say that it is the only option with a positive expected margin of victory against all other options, even though it is not guaranteed to win in any particular trial. This would naturally lead to an entropic argument, similar to Boltzmann (1898), that there is a probability distribution over the options and that it is sharply peaked. However, this assumption is not explicitly mentioned in Coughlin (1992) and no entropic argument is presented.

One methodology for computing a probability distribution over options is the probabilistic Condorcet election, as presented in Wise and Bankes (2015).

Comparison of BLVM with IPV

It is worth noting that the BLVM and IPV provide different probabilistic models of a CDMP. Because the BLVM uses the single f_i function instead of the single u_i function, it also restricts the class of probability functions which can be considered. However, it does restrict them in a different way, because there is in general no 1-to-1 mapping between the P^u probabilities and the P^L probabilities: they simply have different 'shapes'. This is easily seen by the following reductio ad absurdum argument showing that the assumption that they actually can be equated leads to a contradiction.

First, notice that for discrete choice over M options, each u_i distribution has M degrees of freedom, as the increments in Equation 8 can be chosen independently. Similarly, the f_i distribution has M degrees of freedom, because each f_i value can be assigned independently.

Second, consider any two arbitrary options a and b:

$$P_i^u[a:b] - P_i^u[b:a] = \frac{v_i(a:b)}{w_i} = P_i^L[a:b] - P_i^L[b:a]$$

Equation 19: Probabilities related by individual vote

or

$$u_i(a) - u_i(b) = \frac{f_i(a) - f_i(b)}{f_i(a) + f_i(b)}$$

Equation 20 Differences in a-b utilities from a-b f-ratios

Suppose we compare some arbitrary third option, c, with option b, starting with the following equation.

$$u_i(b) - u_i(c) = \frac{f_i(b) - f_i(c)}{f_i(b) + f_i(c)}$$

Equation 21: Difference in b-c utilities from b-c f-ratios

Equation 20 and Equation 21 can simply be added to give the following:

$$u_i(a) - u_i(c) = \frac{f_i(a) - f_i(b)}{f_i(a) + f_i(b)} + \frac{f_i(b) - f_i(c)}{f_i(b) + f_i(c)}$$

Equation 22: Difference in a-c utilities from a-b-c f-ratios

Notice that the left hand side depends only on a and c, while the right hand side depends on a, b and c. Thus, the three values of $f_i(a)$, $f_i(b)$ and $f_i(c)$ cannot be assigned independently, because they are constrained to satisfy the equation. In particular, we can solve this equation for $f_i(b)$ and thus obtain a constraint, which specifies the value of $f_i(b)$, given the other (arbitrary) terms.

We can repeat the process by comparing any other option, d, with b, and obtain the following:

$$u_i(a) - u_i(d) = \frac{f_i(a) - f_i(b)}{f_i(a) + f_i(b)} + \frac{f_i(b) - f_i(d)}{f_i(b) + f_i(d)}$$

Equation 23: Difference in a-d Utilities from a-b-d f-ratios

This is a different constraint, which involves the same $f_i(b)$ but now relates it to a different set of totally arbitrary terms. If we solve for $f_i(b)$, we get a new constraint which specifies a different value of $f_i(b)$. However, it is clearly impossible for one expression to have two different values. Thus we have arrived at a contradiction, so the assumption that the P^u and P^L distributions can be made to coincide is false.

While the two probability models specify different families of probability distributions, neither is a special case of the other. This is easily seen in the case of M discrete choices: each distribution has exactly M degrees of freedom, corresponding to the arbitrary choice of u or f for each option.

Another way of looking at the problem is to consider the number of degrees of freedom in each probability distribution. In an M-by-M matrix of pairwise probabilities, the symmetry relationship that $P[i{:}j] + P[j{:}i] = 1$ means there are only $M(M–1)/2$ free parameters. In other words, given all the probability values in the half-matrix where $i > j$, we can swap the indices and subtract to get the probabilities in the other half-matrix. Along the diagonal, taking $i = k$ and $j = k$ immediately yield $P[k{:}k] = 0.5$

Therefore, we can describe the set of pairwise probabilities consistent with Equation 11 as the following dimensional set, Π:

$$\Pi = \{P[i{:}j] \mid P[i{:}j] > 0.5 \text{ whenever } i > j\}$$

Equation 24: Set of allowed probability distributions

There is a natural expectation that if an actor prefers A over B in the sense that $P[A{:}B] > 0.5$ and prefers B over C, then it should prefer A over C even more strongly. This expectation corresponds to the following constraint. It reduces the volume of Π but not the dimensionality:

$$1 \geq \ P[i{:}j] \geq \max_{i>k>j}\big(\max(P[i{:}k], P[k{:}j])\big)$$

Equation 25: Increasing probabilities from increasing preference

The P^u distribution is determined by the M values of u_i and the P^L distribution by the M values of f_i. Each distribution therefore has only M degrees of freedom, so neither one can represent the full range of pairwise probabilities; they are just too inflexible. Each picks out an M-dimensional subset of Π, and they happen to pick out different subsets.

Both probabilistic voting models are special cases of the Central Position Theorem.

Central Position Theorem

As described in Wise (2010a, 2010b) and published in Jesse (2011), independent proportional voting leads to a surprisingly simple result for the CW, without the restrictive assumptions required for the MVT or the BLVM. This result was termed the Central Position Theorem, or CPT. Having defined the basic terminology of generalized voting, the proof is quite short:

$$V(a{:}b) = \sum_{i=1}^{N} v_i(a{:}b)$$

$$= \sum_{i=1}^{N} w_i \ \big(u_i(a) - u_i(b)\big)$$

$$= \sum_i w_i u_i(a) - \sum_i w_i u_i(b)$$

Equation 26: Net vote from individual proportional votes

We define ‘weighted attractiveness sum’ by the following expression:

$$\zeta(a) = \sum_i w_i u_i(a)$$

Equation 27: Zeta function is weighted attractiveness sum

With this definition, the net group vote is just the difference:

$$V(a{:}b) = \zeta(a) - \zeta(b)$$

Equation 28: Net vote from proportional voting is difference in zeta functions

Notice that we have not postulated a social utility function *de novo* and then found the outcome which maximizes it. Our procedure is akin to the classical model of market equilibrium. The classical market analysis starts with a plausible model of agent interactions, derives the market clearing equilibrium, and only then demonstrates that there is a quantity called ‘total surplus’ and that the many-agent equilibrium maximizes the total surplus.

The classic model of market equilibrium does not postulate *de novo* a function to be maximized but derives one from a dynamic micro-model. Similarly, our micromodel of generalized voting leads to an equilibrium, a balance of influence, and only then demonstrates that it maximizes a particular function.

The CPT states that, for proportional voting over any Ω and any u_i, the dominance relationship is simple:

$$b \rightarrow a \text{ iff } \zeta(b) < \zeta(a)$$

Equation 29: Dominance relationship specified by zeta function

Because the numeric < relationship is transitive and acyclic, this implies that group choice under this model is also transitive and acyclic. Hence, the CW over a set Ω is the central position of that set, defined as that element which maximizes ζ, subject to the same assumptions about closed sets and limits:

$$\theta_{cp} = \underset{\theta\in\Omega}{\operatorname{argmax}}\ \zeta(\theta)$$

Equation 30: Condorcet winner maximizes zeta function

Notice that, the above proof applies whenever Equation 10 holds. It holds both for the deterministic case, where the difference in utility determines the generalized vote, and for the stochastic case, where it determines only the expected value of the vote.

Unlike the MVT or the BLVM, the above proof does not require assumptions of continuity, compactness, convexity, and so on. The CPT thus applies in much more general circumstances, such as the following:

> The above proof applies with a discrete set of options, which cannot be a convex set. Similarly, it can be used without a distance measure, in which case we cannot even define single-peakedness according to Equation 4. Thus we can have a CW, even though the utility function is neither continuous, single-peaked, nor convex. In fact, any finite set of options has a central position, though possibly with ties; it is necessary merely to sort them by $\zeta(\theta)$ and pick the largest.

> The MVT only works in one dimension. However, the CPT also applies to multi-dimensional continuous sets of options, where $\Omega \subseteq R^N$.

> In both unidimensional and multidimensional cases, the utility function can be multi-peaked, yet a CW would still exist at the central position.

> For discrete matching problems, there would usually be no clear measure of distance that would let us even define single peaked or multi-peaked, yet there would still be a CW at the central position.

> It is not required that the actors have similar utility functions, or indeed utility functions of any particular form. The actors could have utility functions driven by a 'naïve' assessment of the direct consequences of implementing an option, or they could be driven by a strategically sophisticated analysis of a multi-stage game conditioned on the choice of θ. The actors might strongly disagree as to the real-world consequences of policies, as is not uncommon in economic policy debates, yet there would still be a CW at the central position.

The CPT has an interesting implication which, at first glance, appears inconsistent with Arrow's Impossibility Theorem (AIT) as presented in Arrow (1951). The AIT starts with the assumption that every actor has transitive, acyclic preferences and uses some fairly mild axioms to determine what acyclic group preferences can be derived. The AIT states that the only consistent solution is to pick one actor as dictator; in other words, it is impossible to define a procedure which aggregates preferences in a reasonable way without introducing cycles.

The CPT implies that, for either interpretation of IPV mentioned above, power weighted voting always leads to acyclic group preferences. This is fully consistent with the AIT because the AIT is a deterministic result, whereas the probabilisic IPV refers only to the expected value of stochastic voting. In particular, it requires a particular class of probabilistic voting functions, P^u, with at least some chance of any option being chosen over any other option by each individual voter, which is contrary to the deterministic, acyclic preferences required for the AIT to hold. Similar comments apply to the relationship between the AIT and the P^L distribution used in Corollary 4.4 of Coughlin (1992).

Further, the AIT requires that no information be available except the preference ordering of actors; the CPT requires additional numeric information on the intensity of that preference, for both deterministic and probabilistic interpretations. This recalls one frequent criticism of the AIT: it cannot produce a unique result because its preconditions rule out all the useful information necessary to get a unique result.

While the CPT is interesting, it is important to recognize its limits. For example, there are many probabilistic voting models consistent with the actor's binary votes which do not lead to the same CW; they generally have no CW at all. One such non-proportional probabilistic voting model uses a cube-root estimator, again with all $w_i = 1$ and all $0 \leq u_i(\theta) \leq 1$:

$$v_i(a{:}b) = w_i \left(u_i(a) - u_i(b)\right)^{1/3}$$

Equation 31: Cube-root voting rule

Further, as with any optimization problem, there are complex issues surrounding the existence of an optimum of ζ, as well as the possibility of multiple equivalent optima.

Finally, even when an optimum of ζ does exist and is unique, the lack of structure in Ω and u_i could make the optimization extremely challenging.

Interdependent Sublinear Voting

While Equation 10 leads to some interesting results, it does not take into account interactions between multiple actors confronting the same choice. It treats them as if they were 'going it alone', as opposed to sharing – or shirking – the burden with like-minded others who are also deciding on what actions to take. If actors do take into account each other's actions, and there are costs or risks to exerting influence, then each actor will be tempted to exert less effort because it knows someone else will make an effort. In other words, there will be a free rider problem. This is a serious objection, as there are many cases where strategic interactions completely change the group preference. The thesis of this paper is that, for a plausible model, the effects are real but the overall group preference is unchanged. The result follows from two competing effects. First, each actor in a coalition is tempted to make less effort, and to avoid the costs of exerting influence, because others are pursuing the same goal. This is the familiar free rider problem and it does indeed reduce the overall level of effort. However, because there are also competing actors pursuing opposing goals, the effort does not decline all the way to zero: each side avoids unnecessarily handing victory to the opponent. So the competition to produce results tends to increase effort on both sides, and the desire not to incur too much cost tends to reduce it. Under this model, the effects cancel each other enough to preserve the group ordering.

We will now present a model of interdependent voting and demonstrate that, in spite of free rider effects, it leads to exactly the same group dominance relationship, and hence has exactly the same Condorcet winner, i.e., the Central Position. Thus, this result can be regarded as fuller statement of the CPT by extending it to behavior not analyzed in Coughlin (1992) or Wise (2010a).

In NCGT, each actor chooses its actions independently, where the payoff to each depends not only on its circumstances and choice but also on the actions of all actors. We now present a simple NCGT model of the expected utility to actors of exerting influence, given the choices of other actors. We no longer postulate a simple strategy for each actor, but derive what an optimal NCGT strategy would be under the assumed model. This necessarily leads to a Nash equilibrium, as each actor takes the behavior of the other actors as a given. It is less myopic than ignoring the other actors entirely, but more myopic than considering the reactions of other actors.

To avoid excessive subscripts when describing the Nash equilibrium quantities, in this section we shorten $s(a{:}b)$ to simply X. The members of the coalition $c(a{:}b)$ are indexed by the subscript i. Their votes for a will be denoted as x_i, so $X = \sum_i x_i$

$$X = \sum_i x_i = s(a{:}b) = \sum_{k \in c(a:b)} v_k\,(a{:}b)$$

Equation 32: Concise notation for coalition strength

All the votes for a except i's will be denoted $X_i = X - x_i$.

Similarly, we will shorten $s(b{:}a)$ to simply Y. The coalition for b will be indexed by j, their votes as y_j, and $y_j = Y - y_j$. We shorten $u_i\,(a)$ to a_i and $u_i\,(b)$ to b_i; similarly for b_j and a_j.

We model the probability of either outcome prevailing as not deterministic but probabilistic. While the discrete counting after a committee vote may yield a deterministic outcome, the informal exertion of backroom influence before the vote is generally more uncertain, unless one side has overwhelming influence. Similar considerations apply to the other forms of informal influence mentioned earlier.

Going back to at least the World War II work of Weiner (1948 and 1949), modern control theory frequently treats the generalized 'cost', or disutility, of control as quadratic in the level of control exerted. It is assumed that disutility is a more generic measure than financial cost. The notion of quadratic disutility is merely an analytically convenient heuristic to approximate the idea that there are definite limits to how much control can be exerted, and generalized cost increases rapidly as those limits are approached. The notion of quadratic disutility has been extremely successful in millions of real-world applications of control theory over more than half a century; this approach is supported by the rich body of theory for linear-quadratic-Gaussian control problems.

We follow this tradition and model the disutility to actor i of exerting influence as a quadratic function of the influence exerted:

$$d_i(v_i) = c_i v_i^2$$

Equation 33: Disutility of exerting influence

Notice that the disutility of exerting influence is not the same thing as the influence itself. For example, the influence exerted in a military context might be measured in terms of 'boots on the ground', while the disutility of influence might be measured in terms of dollar expenditure to support an overseas expeditionary force, loss of votes from angered constituents, etc.

We do not expect the specific linear form of our result to hold for arbitrary disutility functions.

As mentioned earlier, legislative committees use a deterministic decision rule where a margin of even one vote fully determines the outcome. However, we will focus on generalized voting, or exertion of informal influence, where the probability of either outcome depends on the relative strengths of each coalition:

$$P_a = \frac{X}{X+Y}$$

$$P_b = \frac{Y}{X+Y}$$

Equation 34: Probabilities as ratios of coalition strength

Neglecting the disutility of exerting influence, the expected utility to a given actor depends on the probabilities of each outcome:

$$E[u_i] = a_i P_a + b_i P_b$$

Equation 35: Expected utility is probability-weighted average

Combining the expected utility of the outcome with quadratic disutility yields the following equation for the overall expected utility to actor i:

$$u_i(v_i) = a_i \frac{X}{X+Y} + b_i \frac{Y}{X+Y} - c_i v_i^2$$

Equation 36: Overall utility is expected gain from influence minus disutility of influence

Without loss of generality, we focus on an actor, i, that favors option a. As mentioned earlier, votes favoring option a will be denoted by x_i, to avoid a proliferation of conditionals embedded inside equations.

$$u_i(x_i) = a_i \frac{(X_i + x_i)}{(X_i + x_i) + Y} + b_i \frac{Y}{(X_i + x_i) + Y} - c_i x_i^2$$

Equation 37: Overall utility for an actor favoring a

If we treat Y and X_i as constants, we can set the slope to zero, $\partial u_i / \partial x_i = 0$, to get the optimal level of influence for actor i to exert:

$$x_i = \frac{a_i - b_i}{2c_i} \frac{Y}{(X_i + x_i + Y)^2}$$

Equation 38: Optimum level of influence to exert favoring a

Comparison with Equation 10 suggests that, to ensure comparability, we set the cost factor so that the relative levels of influence exerted are comparable, i.e. $\lambda w_i = 1/(2c_i)$, which yields the following:

$$x_i = w_i(a_i - b_i)\frac{\lambda Y}{(X_i + x_i + Y)^2}$$

Equation 39: Interdependent, sublinear voting

As expected, the optimal vote by each actor now depends on all the other actors' votes. In particular, the more actors support a, the larger will be X_i and the lower will be i's exertion of influence, as we expect from free riders. Clearly, the votes of the actors are interdependent and cannot be specified separately by an independent-voting rule such as those presented earlier.

The level of influence exerted is not strictly linear in the stakes $a_i - b_i$, as there is a divisor which increases quadratically in x_i. So we expect the total level of influence exerted to be somewhat less than a strictly linear proportionality suggested by Equation 10.

Similarly,

$$y_j = w_j(b_j - a_j)\frac{\lambda X}{(Y_j + y_j + X)^2}$$

Equation 40: Optimum level of influence to exert favoring b

Nash Equilibrium Voting

To determine the voting pattern, we need to find the Nash equilibrium which results when each actor independently chooses its optimum level of influence as per Equations 39 & 40. First, we define two useful scaling constants:

$$X_0 = \sum_i w_i(a_i - b_i)$$

$$Y_0 = \sum_j w_j(b_j - a_j)$$

Equation 41: Scaling constants for net vote

We can restate the dominance relation under independent proportional voting as:

$$V(a{:}b)>0 \text{ iff } X_0>Y_0$$

Equation 42: Dominance under proportional voting

Equivalently,

$$b\to a \text{ iff } X_0>Y_0$$

Equation 43: Dominance relationship from ccaling constants

We now examine the corresponding relationships at Nash equilibrium. As noted earlier, X_i in Nash equilibrium is defined to be X-x_i, so the value X is X_i+x_i at Nash equilibrium, independent of i; similarly for Y. Thus,

$$x_i = \lambda w_i(a_i - b_i)\frac{Y}{(X + Y)^2}$$

$$y_j = \lambda w_j(b_j - a_j)\frac{X}{(Y + X)^2}$$

Equation 44: Individual votes at Nash Equilibrium

Summing over i:

$$X = \sum_i x_i$$

$$= \sum_i \lambda w_i(a_i - b_i)\frac{Y}{(X + Y)^2}$$

$$= \lambda X_0 \frac{Y}{(X + Y)^2}$$

Equation 45: Net votes for a in Nash Equilibrium

Similarly,

$$Y = \lambda Y_0 \frac{X}{(Y + X)^2}$$

Equation 46: Net votes for b in Nash Equilibrium

Hence,

$$\frac{X_0}{Y_0} = \left(\frac{X}{Y}\right)^2$$

Equation 47: Relationship between independent and interdependent net votes. Note that this relationship holds, regardless of the as-yet-undetermined scaling parameter λ.

Thus, if the ratio $1.21 = X_0/Y_0$ for independent proportional voting, then $1.1 = X/Y$ for interdependent sublinear voting. Again, the free rider problem means that actors exert less influence, so that option a's margin of victory over b is reduced, but never reversed:

$$X \lesseqgtr Y \text{ iff } X_0 \lesseqgtr Y_0$$

Equation 48: Identical dominance for independent and interdependent voting

This shows that the dominance relationship under interdependent, sublinear voting is exactly identical to that under independent, proportional voting. Hence, the Condorcet winner is the same, again without requiring special conditions on Ω or u_i except for the technical conditions necessary to ensure a maximum of $\zeta(\theta)$.

For some purposes, it is necessary to know not just the ratio X/Y but their exact values. Define the probability of either option failing:

$$\lambda_a = \frac{Y}{X+Y}$$

$$\lambda_b = \frac{X}{X+Y}$$

Equation 49: Success probabilities in Nash Equilibrium

So,

$$x_i = \lambda\, \lambda_a\ w_i\, (a_i - b_i)$$

$$y_j = \lambda\, \lambda_b\ w_j\, (b_j - a_j)$$

Equation 50: Individual votes with scaling factors

Summing,

$$X = \lambda\, \lambda_a\ X_0$$

$$Y = \lambda\, \lambda_b\ Y_0$$

Equation 51: Net votes with scaling factors

So,

$$\frac{\lambda_a}{\lambda_b} = \frac{\sqrt{Y_0}}{\sqrt{X_0}}$$

In other words, $\lambda_a = z\sqrt{Y_0}$ and $\lambda_b = z\sqrt{X_0}$ for some scale factor z. The scalar z is set by the requirement that $\lambda_a + \lambda_b = 1$, so

$$\lambda_a = \frac{\sqrt{Y_0}}{\sqrt{Y_0} + \sqrt{X_0}}$$

$$\lambda_b = \frac{\sqrt{X_0}}{\sqrt{Y_0} + \sqrt{X_0}}$$

It is natural to set the scale λ so that

$$X_0 + Y_0 = X + Y$$

$$= \lambda\, \lambda_a\ X_0 + \lambda\, \lambda_b\ Y_0$$

Equation 54: Relationship between overall scale factor and Nash Equilibrium votes

So the overall scale factor λ is the following:

$$\lambda = \frac{X_0 + Y_0}{\lambda_a X_0 + \lambda_b Y_0}$$

Equation 55: Overall scale factor

Conclusion

The main purpose of this paper is to explore several CDMPs and demonstrate that the independent voting model leads to the exact same collective decision as does the interdependent voting model. We reviewed the first two clauses of the Central Position Theory and extended it with a third clause:

> Under independent proportional voting, for a very wide range of Ω sets and u_i functions, the group preference relationship is acyclic and hence has a Condorcet winner.

> That CW can be found by maximizing the Benthamite social utility function, ζ.

> If independent proportional voting is replaced by the Nash equilibrium of interdependent voting with quadratic cost, the group preference is not changed, and the CW of the new CDMP can still be found by maximizing ζ.

One result of the CPT was shown to be that some probabilistic models of acyclic voter preferences do give acyclic group preferences under weighted binary voting with majority rule. This demonstrates one of the connections between the CPT and probabilistic voting theory.

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About the Author

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About the Project

KAPSARC is developing the KAPSARC Toolkit for Behavioral Analysis (KTAB), an open source software platform, to support modeling and analysis of collective decision making processes (CDMPs). Within our research, KTAB is intended to be the standard platform for analyzing bargaining problems, generalized voting models and policy decision making. It is our intent to use KTAB to assemble the building blocks for a broad class of CDMPs. Typical models in KTAB will draw on the insights of subject matter experts regarding decision makers and influencers in a methodical, consistent manner and will then assist researchers to identify feasible outcomes that are the result of CDMPs.

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